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Published in:
Methods of Operations Research

Publication date:
1981

RISK SENSITIVITY OF BARGAINING SOLUTIONS

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ABSTRACT

Bargaining situations are considered for pairs of bargainers, who may agree upon an element of a set of risky and riskless feasible outcomes, or who may disagree, in which case a prescribed disagreement outcome is the result. Such situations can be reduced to a bargaining game as introduced by J. Nash. We deal with the question, whether it is advantageous or disadvantageous for a bargainer to have a more risk averse opponent in such a bargaining situation. Special attention is given to the risk aspects of the generalized Nash bargaining solutions, the Kalai-Rosenthal solution and the Kalai-Smorodinsky solution.

Keywords: Bargaining Solution, Risk Aversion, Bargaining Game.

1. INTRODUCTION

In this paper we study bargaining situations, in which pairs of bargainers (players) are involved, who are faced with a non-empty set A of feasible alternatives. If the bargainers agree upon one of these alternatives, or upon a finite lottery of those alternatives, then this will be the result of the bargaining. In the case that no unanimous agreement is reached a disagreement outcome $\underline{a} \in A$, known to the bargainers, results. The result which a player can achieve in such a bargaining situation will, in general, not only depend on his own preferences but also on the preferences of the other player. Especially, the question whether it is more advantageous to have a more risk averse opponent is the problem of this paper. Inspiring for us were the recent papers of Kihlstrom, Roth and Schmeidler (1980) and of Roth and Rothblum (1980).

The organization of the paper is as follows. In section 2 we define bargaining situations and show how to reduce them to bargaining games as introduced by J.F. Nash (1950). In section 3 we deal with the characterization of risk aversion. The risk sensitivity of the generalized Nash solutions is studied in section 4 and the risk sensitivity of the Kalai-Rosenthal and Kalai-Smorodinsky solutions in section 5. We conclude in section 6 with some remarks and a conjecture.

2. BARGAINING SITUATIONS AND BARGAINING GAMES

Let A be a non-empty compact subset of $\mathbb{R}^n$ and let L(A) be the set of finite lotteries over A. For $a_1, a_2, \ldots, a_m \in A$ and non-negative numbers $p_1, p_2, \ldots, p_m$ with $\sum_{i=1}^m p_i = 1$, we denote the lottery, which chooses a_i with probability p_i $(i = 1, \ldots, m)$, by $[p_1, a_1; p_2, a_2; \ldots; p_m, a_m]$ or by $[p_i, a_i]_{i=1}^m$. An element a of A will be identified with the lottery [1,a], which chooses a with probability 1. We are interested in functions $u: L(A) \to \mathbb{R}$ with the following properties:

(2.1) the restriction of u to A is continuous

(2.2) the restriction of u to A is concave i.e. if a convex combination $c = \sum_{i=1}^m \alpha_i a_i$ of elements of A is also an element of A, then $u(c) \geq \sum_{i=1}^m \alpha_i u(a_i)$

(2.3) for each lottery $\ell = [p_i, a_i]_{i=1}^m$ we have $u(\ell) = \sum_{i=1}^m p_i u(a_i)$.

The class of these functions will be denoted by U(A). In the following we suppose that the preferences of bargainers on L(A) can be described by elements of U(A). [cf. Fishburn (1977)].

DEFINITION 2.1. A <u>bargaining situation</u> is a quadruplet $\langle A, \underline{a}, u_1, u_2 \rangle$, where A is a non-empty compact subset of $\mathbb{R}^n$, $\underline{a} \in A$ and $u_1, u_2 \in U(A)$. A is called the <u>set of riskless alternatives</u>, $\underline{a}$ the <u>disagreement outcome</u> and u_1 and u_2 the <u>utility functions</u> of the bargainers 1 and 2, respectively. The elements of $L(A)\backslash A$ are called <u>risky alternatives</u> of the bargaining situation. The family of all bargaining situations will be denoted by BS.

We will relate bargaining situations to bargaining games, introduced by Nash (1950). Here is the definition.

DEFINITION 2.2. A <u>bargaining game</u> is a pair $\langle S, d \rangle$, where S is a non-empty convex compact subset of $\mathbb{R}^2$ and d is an element of S.

The family of all bargaining games is denoted by BG. An

$<S,d> \in BG$ corresponds, intuitively, to a situation, in which two players are involved, and where the i-th coordinate d_i of d is the utility of player i, attained if the players do not cooperate, and where S contains all the attainable utility pairs when they do cooperate.

Now, let $\Gamma = <A,\underline{a},u_1,u_2>$ be a bargaining situation. Let S_Γ be the convex hull of the set $\{(u_1(a),u_2(a)) \in \mathbb{R}^2 ;\ a \in A\}$ and $d_\Gamma = (u_1(\underline{a}),u_2(\underline{a}))$. Then $<S_\Gamma,d_\Gamma> \in BG$ will be called the <u>bargaining game corresponding to the bargaining situation</u> Γ. Note that S_Γ is compact because $\{(u_1(a),u_2(a)) \in \mathbb{R}^2 ;\ a \in A\}$ is the image of the compact set A under the continuous map (u_1,u_2).

For a non-empty compact convex subset S of $\mathbb{R}^2$, the set $\{p \in S;$ for each $s \in S$ with $s \geq p$, we have $s = p\}$ is called the <u>Pareto set</u> of S and is denoted by P(S); the <u>weak Pareto set</u>

$$\{w \in S;\ \text{for each } s \in S \text{ with } s \geq w, \text{ we have } s_1 = w_1 \text{ or } s_2 = w_2\}$$

is denoted by $W(S)$. For further use we note that the Pareto set P(S) of S is the graph of the decreasing concave function $\pi_S\colon\ [\alpha_S\ ,\ \beta_S] \to \mathbb{R}$ where $\alpha_S := \min\{p_1; (p_1,p_2) \in P(S)\}$, $\beta_S := \max\{p_1; (p_1,p_2) \in P(S)\}$ and, for $x \in [\alpha_S,\beta_S]$, $\pi_S(x)$ is the second coordinate of the (unique) Pareto point with first coordinate x. We call π_S the <u>Pareto function</u> of S.

DEFINITION 2.3. A map $\phi\colon BG \to \mathbb{R}^2$ is called a <u>bargaining solution</u> for bargaining games if the following properties hold:

(2.4) $\phi(S,d) \in P(S)$ for each $<S,d> \in BG$ (<u>Pareto optimality</u>)

(2.5) $\phi(S,d) \geq d$ for each $<S,d> \in BG$ (<u>Individual rationality</u>)

Each bargaining solution $\phi\colon BG \to \mathbb{R}^2$ induces a bargaining solution $\tilde{\phi}\colon BS \to \mathbb{R}^2$ for bargaining situations, where for each $\Gamma = <A,\underline{a},u_1,u_2> \in BS$, $\tilde{\phi}(\Gamma) := \phi(S_\Gamma,d_\Gamma)$. Instead of $\tilde{\phi}$ we will in the following also write ϕ. In the sequel, subclasses of BS will play a role. We introduce them here

$$BSC := \{\Gamma = <A,\underline{a},u_1,u_2> \in BS;\ A \text{ is convex}\}$$
$$BSR := \{\Gamma = <A,\underline{a},u_1,u_2> \in BS;\ P(S_\Gamma) \subset \{(u_1(a),u_2(a)) \in \mathbb{R}^2; a \in A\}\}$$
$$BSI := \{\Gamma = <A,\underline{a},u_1,u_2> \in BS;\ d_\Gamma \notin W(S_\Gamma)\}$$
$$BSW := \{\Gamma = <A,\underline{a},u_1,u_2> \in BS;\ d_\Gamma \in W(S_\Gamma)\}$$

BSR consists of those bargaining situations Γ, for which the Pareto-points of S_Γ correspond to riskless alternatives. It is easy to show that $BSC \subset BSR$.

BSW (BSI) consists of the bargaining situations Γ for which in the corresponding bargaining game the disagreement point d_Γ is (not) a point of the weak Pareto boundary.

3. RISK AVERSION

Let A be a non-empty compact subset of $\mathbb{R}^n$, the set of riskless alternatives for decision makers. Suppose we have two decision makers, one with utility function $u \in U(A)$ and the other with utility functions $v \in U(A)$. An old problem is, how to compare the aversion to risk of the two decision makers. Inspired by the pioneering work of Pratt (1964) and Arrow (1971), Yaari (1969) proposed to tackle the problem with the aid of acceptance sets. For $u \in U(A)$ and $a \in A$, the <u>acceptance set</u> $A_u(a)$ is the set of lotteries, which are at least as acceptable as the riskless outcome a i.e.

$$A_u(a) := \{\ell \in L(A);\ u(\ell) \geq u(a)\}$$

Yaari called decision maker v more risk averse than decision maker u (notation $v\,Y\,u$) if $A_v(a) \subset A_u(a)$ for all $a \in A$. It is clear that Y is not a complete relation, because $v\,Y\,u$ implies

(3.1) If for $a,b \in A$ we have $v(b) \geq v(a)$, then $u(b) \geq u(a)$.

Unfortunately not necessarily holds:

(3.2) If $u(b) = u(a)$, then $v(b) = v(a)$.

Kihlstrom and Mirman (1974) found a mathematical convenient characterization of risk aversion in case condition (3.2) holds. They proved: If u and v are elements of U(A) satisfying (3.2), then $v\,Y\,u$ iff v is an increasing concave transformation of u.

We want to make a slightly different approach to the problem, leading to a nicer analogue of the Kihlstrom-Mirman result. For $u \in U(A)$ and $a \in A$, we introduce the <u>preference set</u> $P_u(a)$, the set of lotteries which are preferred to the riskless outcome a. In formula:

$$P_u(a) := \{\ell \in L(A);\ u(\ell) > u(a)\}.$$

Let $u,v \in U(A)$. Then we call v <u>more risk averse than</u> u (notation $v\,MR\,u$) if $P_v(a) \subset P_u(a)$ for all $a \in A$.

Then we have

LEMMA 3.1. Suppose $u,v \in U(A)$ and $v\,MR\,u$. Then for all $a,b \in A$

(i) if $v(b) > v(a)$, then $u(b) > u(a)$

(ii) if $u(a) \leq u(b)$, then $v(a) \leq v(b)$

(iii) if $u(a) = u(b)$, then $v(a) = v(b)$.

PROOF. Suppose $v(b) > v(a)$. Then $b \in P_v(a) \subset P_u(a)$. So $u(b) > u(a)$ and (i) is proved. (ii) follows immediately from (i) and for (iii) we note that (i) implies: if $v(a) \neq v(b)$, then $u(a) \neq u(b)$. □

The relation MR can be characterized as follows.

THEOREM 3.2. Let $u, v \in U(A)$. Then the following two assertions are equivalent:

(i) $v\,\mathrm{MR}\,u$

(ii) there exists a non-decreasing concave transformation $k\colon u(A) \to \mathbb{R}$ such that $v = k \circ u$.

PROOF. (a) Suppose (i). In view of lemma 3.1(iii), there exists a function $k\colon u(A) \to \mathbb{R}$ such that $k(u(a)) = v(a)$ for all $a \in A$. In view of (ii) of lemma 3.1, we may conclude that k is non-decreasing on $u(A)$. Now we want to show that $k\colon u(A) \to \mathbb{R}$ is concave. Let $u(a)$ be the convex combination $\sum_{i=1}^m p_i u(a_i)$ of $u(a_1), \ldots, u(a_m)$. We have to prove that $k(u(a)) \geq \sum_{i=1}^m p_i k(u(a_i))$ or that

$$v(a) \geq \sum_{i=1}^m p_i v(a_i) = v([p_i, a_i]_{i=1}^m).$$

If $v(a) < v([p_i, a_i]_{i=1}^m)$, then $[p_i, a_i]_{i=1}^m \in P_v(a) \subset P_u(a)$, which implies the contradiction

$$u(a) < u([p_i, a_i]_{i=1}^m) = \sum_{i=1}^m p_i u(a_i) = u(a).$$

So k is concave and non-decreasing on $u(A)$, proving (ii).

(b) Let $k\colon u(A) \to \mathbb{R}$ be a non-decreasing concave function, satisfying $v = k \circ u$. For the proof of (ii) $\Rightarrow$ (i) we need the extension $\tilde{k}\colon \mathrm{conv}(u(A)) \to \mathbb{R}$ of k to the convex hull of $u(A)$, which is defined as follows: $\tilde{k}(x) = k(x)$ if $x \in u(A)$, and for $x \in (\mathrm{conv}\ u(A)) \backslash u(A)$: $\tilde{k}(x) = \alpha k(y) + (1-\alpha) k(z)$ where

$$y := \max\{u(c);\ c \in A,\ u(c) < x\}$$

$$z := \min\{u(c);\ c \in A,\ u(c) > x\}$$

and $\alpha \in (0,1)$ is such that $x = \alpha y + (1-\alpha) z$.

[The existence of max and min follows from the compactness of A and the continuity of u.]

It is straightforward to show that $\tilde{k}$ is also non-decreasing and concave. Now take $a \in A$, $\ell = [p_i, a_i]_{i=1}^m \in L(A)$ with $\ell \in P_v(a)$. Then

$$v(a) < v(\ell) = \sum_{i=1}^m p_i v(a_i), \text{ which is equivalent to}$$

$k(u(a)) < \sum_{i=1}^m p_i k(u(a_i))$. Since $\tilde{k}$ is concave, we obtain: $\tilde{k}(u(a)) < \sum_{i=1}^m p_i \tilde{k}(u(a_i)) \leq \tilde{k}(\sum_{i=1}^m p_i u(a_i))$, and since $\tilde{k}$ is non-decreasing: $u(a) < \sum_{i=1}^m p_i u(a_i) = u(\ell)$. So $\ell \in P_u(a)$. Hence $P_v(a) \subset P_u(a)$ for each $a \in A$. □

In the following the notion of risk sensitivity of solutions is central.

DEFINITION 3.3. Let B^* be a non-empty subset of the set of bargaining situations BS. A bargaining solution ϕ: BS $\to \mathbb{R}^2$ is called <u>risk sensitive on</u> B^*, if for all $\Gamma = \langle A,\underline{a},u_1,u_2\rangle \in B^*$ and all $v_2 \in U(A)$ with $v_2 \, MR \, u_2$, we have for $\hat{\Gamma} = \langle A,\underline{a},u_1,v_2\rangle$ that $\phi_1(\hat{\Gamma}) \geq \phi_1(\Gamma)$.

If ϕ is risk sensitive on B^*, then for each bargaining situation in B^* it is advantageous for player 1 when his opponent is replaced by a more risk averse one.

In the following theorem we give a first result about risk sensitivity, which holds for all bargaining solutions.

THEOREM 3.4. Let ϕ: BS $\to \mathbb{R}^2$ be a bargaining solution. Then ϕ is risk sensitive on the subclass BSW.

PROOF. Let $\Gamma = \langle A,\underline{a},u_1,u_2\rangle \in$ BSW and $\hat{\Gamma} = \langle A,\underline{a},u_1,v_2\rangle$ with $v_2 = k \circ u_2$, where k is a non-decreasing concave function. We have to show that $\phi_1(\hat{\Gamma}) \geq \phi_1(\Gamma)$. Let π: $[\alpha_{S_\Gamma},\beta_{S_\Gamma}] \to \mathbb{R}$ be the Pareto function of S_Γ. We distinguish three cases:

(i) Suppose $d_\Gamma \in P(S_\Gamma)$. Then, by (2.4) and (2.5), $d_\Gamma = \phi(S_\Gamma,d_\Gamma)$. Now $d_{\hat{\Gamma}} = ((d_\Gamma)_1,k((d_\Gamma)_2))$. By (2.5), $\phi_1(\hat{\Gamma}) \geq (d_{\hat{\Gamma}})_1 = (d_\Gamma)_1 = \phi_1(S_\Gamma,d_\Gamma)$.

(ii) Suppose $(d_\Gamma)_1 = \beta_{S_\Gamma}$. Then $\phi_1(S_{\hat{\Gamma}},d_{\hat{\Gamma}}) \geq (d_{\hat{\Gamma}})_1 = (d_\Gamma)_1 = \phi_1(S_\Gamma,d_\Gamma)$ because $\phi(S_\Gamma,d_\Gamma) = (\beta_{S_\Gamma},\pi(\beta_{S_\Gamma}))$.

(iii) Suppose $(d_\Gamma)_2 = \pi(\alpha_{S_\Gamma})$. Then $\phi(S_\Gamma,d_\Gamma) = (\alpha_{S_\Gamma},\pi(\alpha_{S_\Gamma}))$. Let $\hat{\pi}$: $[\alpha_{S_{\hat{\Gamma}}},\beta_{S_{\hat{\Gamma}}}] \to \mathbb{R}$ be the Pareto function of $S_{\hat{\Gamma}}$. Since k is non-decreasing, we have $\alpha_{S_{\hat{\Gamma}}} \geq \alpha_{S_\Gamma}$. But then, by (2.4),

$$\phi_1(S_{\hat{\Gamma}},d_{\hat{\Gamma}}) \geq \alpha_{S_{\hat{\Gamma}}} \geq \alpha_{S_\Gamma} = \phi_1(S_\Gamma,d_\Gamma). \quad \square$$

4. RISK SENSITIVITY OF THE GENERALIZED NASH SOLUTIONS

For each $t \in (0,1)$, the corresponding generalized Nash solution F^t: BG $\to \mathbb{R}^2$ assigns to $(S,d) \in$ BG the unique point $F^t(S,d) = (F_1^t(S,d),F_2^t(S,d))$ of $P(S)$, for which

$$(F_1^t(S,d)-d_1)^t(F_2^t(S,d)-d_2)^{1-t} = \max_{p \in P(S)} (p_1-d_1)^t(p_2-d_2)^{1-t}.$$

Each solution F^t has the following nice properties (cf. Harsanyi and Selten (1972), Kalai (1978) and Roth (1979)): (i) Pareto optimality, (ii) Individual rationality, (iii) Independence of irrelevant alternatives, (iv) Independence of equivalent utility representations. $F^{\frac{1}{2}}$ is called the symmetric Nash solution, and was introduced in Nash (1950).

Kihlstrom, Roth and Schmeidler (1980) proved that the solution $F^{\frac{1}{2}}$ is risk sensitive on the class $BSC \cap BSI$ of bargaining situations.

In this section we will prove that for each $t \in (0,1)$, the bargaining solution F^t is risk sensitive on BSR ($\supset$ BSC). For this purpose we need a characterization of $F^t(S,d)$ on $P(S)$, described in the following proposition. First we introduce, for $(S,d) \in BG$, the slope multifunction $\psi\colon [\alpha_S, \beta_S] \to [-\infty, 0]$, where $\psi(\alpha_S) = [D_r\pi_S(\alpha_S), 0]$, $\psi(\beta_S) = [-\infty, D_\ell\pi_S(\beta_S)]$ and $\psi(u) = [D_r\pi_S(u), D_\ell\pi_S(u)]$ if $\alpha_S < u < \beta_S$. Here D_r and D_ℓ stand for right derivative and left derivative, respectively, and π_S is the Pareto function of S. Note that ψ assigns to each $u \in [\alpha_S, \beta_S]$ the set of slopes of all supporting lines on S with supporting point $(u, \pi_S(u))$. Note further that ψ is a monotone non-increasing multifunction (i.e. for all $u,v \in [\alpha_S, \beta_S]$: if $u < v$, $x \in \psi(u)$, $y \in \psi(v)$, then $x \geq y$), that $\psi(u)$ is a convex set for each u and that for each $r \in [-\infty, 0]$, there is an $u \in [\alpha_S, \beta_S]$ with $r \in \psi(u)$. In proposition 4.1 also the increasing function $\lambda_t\colon (\max(d_1, \alpha_S), \beta_S] \to \mathbb{R}$, defined by $\lambda_t(u) = t(t-1)^{-1}(\pi_S(u) - d_2)(u - d_1)^{-1}$, plays a role.

PROPOSITION 4.1. Let $t \in (0,1)$, $(S,d) \in BG$, $d \notin W(S)$, $z \in P(S)$, and $p \in P(S)$ with $p > d$. Then

(i) $z = F^t(S,d)$ iff $\lambda_t(z_1) \in \psi(z_1)$

(ii) If $\lambda_t(p_1) \leq D_\ell\pi_S(p_1)$, then $F_1^t(S,d) \geq p_1$

(iii) If $\lambda_t(p_1) \geq D_r\pi_S(p_1)$, then $F_1^t(S,d) \leq p_1$.

PROOF. (a) Suppose $z = F^t(S,d)$. Let $\gamma := (z_1-d_1)^t(z_2-d_2)^{1-t}$. Then $\{z\} = S \cap T$ where $T := \{(x_1,x_2) \in \mathbb{R}^2;\ x \geq d,\ (x_1-d_1)^t(x_2-d_2)^{1-t} \geq \gamma\}$. T is a convex set because T is the epigraph of the strict convex function $f\colon [d_1, \infty)$ defined by $f(x_1) = d_2 + \gamma^{(1-t)^{-1}}(x_1-d_1)^{t(t-1)^{-1}}$. Since f is differentiable, there is a unique separating line for S and T with slope

$$f'(z_1) = \gamma^{(1-t)^{-1}} t(t-1)^{-1}(z_1-d_1)^{(t-1)^{-1}} =$$

$$= t(t-1)^{-1}(z_2-d_2)(z_1-d_1)^{-1} = \lambda_t(z_1).$$

This implies $\lambda_t(z_1) \in \psi(z_1)$. For the implication to the left in (i), note that from the mentioned monotonicity properties of ψ and λ_t, it follows that there is at most one z with $\lambda_t(z_1) \in \psi(z_1)$.

(b) The properties (ii) and (iii) follow from (i) and the monotonicity of λ_t and ψ. □

THEOREM 4.2. For each $t \in (0,1)$, F^t is risk sensitive on BSR.

PROOF. Take $t \in (0,1)$. In view of theorem 3.4 it is sufficient to show that F^t is risk sensitive on BSR $\cap$ BSI. Let $\Gamma = \langle A, \underline{a}, u_1, u_2 \rangle \in$ BSR $\cap$ BSI and $z = F^t(S_\Gamma, d_\Gamma)$. Let $\hat{\Gamma} = \langle A, \underline{a}, u_1, v_2 \rangle$ with $v_2 = k \circ u_2$ and k is non-decreasing and concave. If k is not increasing in a neighbourhood of z_2, then $\alpha_{\hat{\Gamma}} \geq z_1$ which implies $F_1^t(S_{\hat{\Gamma}}, d_{\hat{\Gamma}}) \geq \alpha_{\hat{\Gamma}} \geq z_1 = F_1^t(S_\Gamma, d_\Gamma)$, so the theorem is proved for this case. Now suppose that k is increasing in a neighbourhood of z_1. Note that $d_2 < z_2$, because $\Gamma \in$ BSI. Since F^t is independent of equivalent utility representations,we may suppose w.l.o.g. that $k(d_2) = d_2$ and $k(z_2) = z_2$. Then the concavity of k implies

$$D_r k(z_2) \leq D_\ell k(z_2) \leq (z_2 - d_2)^{-1}(k(z_2) - k(d_2)) = 1.$$

Let $\hat{\pi}$ be the Pareto function of $S_{\hat{\Gamma}}$. Then

$$D_\ell \hat{\pi}(z_1) = D_\ell(k \circ \pi_S)(z_1) = D_r k(z_2) \cdot D_\ell \pi_S(z_1) \geq D_\ell \pi_S(z_1) \qquad (4.1)$$

Then, by proposition 4.1(i): $\hat{\lambda}_t(z_1) = \lambda_t(z_1) \in \psi(z_1) = [D_r \pi_S(z_1), D_\ell \pi_S(z_1)]$. So, by (4.1) $\hat{\lambda}_t(z_1) \leq D_\ell \pi_S(z_1) \leq D_\ell \hat{\pi}(z_1)$, which implies in view of proposition 4.1(ii): $F_1^t(S_{\hat{\Gamma}}, d_{\hat{\Gamma}}) \geq z_1 = F_1^t(S_\Gamma, d_\Gamma)$. □

Jansen en Tijs (1980) considered the bargaining solutions F^0: BG $\to \mathbb{R}^2$ and F^1: BG $\to \mathbb{R}^2$. For each $(S,d) \in$ BG, $F^1(S,d)$ is the point in $\{p \in P(S);\ p \geq d\}$ with maximal first coordinate and $F^0(S,d)$ the point in $\{p \in P(S);\ p \geq d\}$ with maximal second coordinate. These bargaining solutions share many of the features of the generalized Nash solutions e.g. the four mentioned properties in the beginning of this section are satisfied by F^0 and F^1. Also a similar result w.r.t. risk posture holds as we see in the following theorem. The proof of this theorem is straightforward and can be found in Peters (1981).

THEOREM 4.3. F^0 and F^1 are risk sensitive on BSR.

In Roth and Rothblum (1980), bargaining games $\Gamma = \langle A, \underline{a}, u_1, u_2 \rangle$ are studied, where not all Pareto points of S_Γ are utility pairs of riskless outcomes. In order to formulate their main result we need some terminology. First note that for each $p \in P(S_\Gamma)$ there exist a_1 and a_2 in A and $\alpha \in (0,1)$ such that $p = \alpha(u_1(a_1), u_2(a_1)) + (1-\alpha)(u_1(a_2), u_2(a_2))$, and such that there is no point $a_3 \in A \backslash \{a_1, a_2\}$ such that $(u_1(a_3), u_2(a_3))$ is a convex combination of $(u_1(a_1), u_2(a_1))$ and $(u_1(a_2), u_2(a_2))$. [If $p = (u_1(a), u_2(a))$

for some $a \in A$, then $a_1 = a_2 = a$.]
Roth and Rothblum call the point p (u_1,u_2)-supported by a_1 and a_2. Further, p is called favourably (u_1,u_2)-supported if $u_2(a_1) \geq u_2(\underline{a})$ and $u_2(a_2) \geq u_2(\underline{a})$, and unfavourably (u_1,u_2)-supported if otherwise. Let $\Gamma \in BSI$ and $\hat{\Gamma} = \langle A,\underline{a},u_1,k \circ u_2\rangle$ where k is an increasing concave function. The main result of Roth and Rothblum is: If $F^{\frac{1}{2}}(S_\Gamma,d_\Gamma)$ is favourably (u_1,u_2)-supported, then $F_1^{\frac{1}{2}}(S_{\hat{\Gamma}},d_{\hat{\Gamma}}) \geq F_1^{\frac{1}{2}}(S_\Gamma,d_\Gamma)$; if $F^{\frac{1}{2}}(S_{\hat{\Gamma}},d_{\hat{\Gamma}})$ is unfavourably $(u_1,k \circ u_2)$-supported, then $F_1^{\frac{1}{2}}(S_{\hat{\Gamma}},d_{\hat{\Gamma}}) \leq F_1^{\frac{1}{2}}(S_\Gamma,d_\Gamma)$.

This result can be extended to the generalized Nash solutions and to F^0 and F^1. In Peters (1981) the following theorem is proved:

THEOREM 4.4. Let $\Gamma = \langle A,\underline{a},u_1,u_2\rangle \in BS$ and $\hat{\Gamma} = \langle A,\underline{a},u_1,k \circ u_2\rangle \in BS$ with k a non-decreasing concave function. Let $t \in [0,1]$. Then:

(i) If $F^t(S_\Gamma,d_\Gamma)$ is favourably (u_1,u_2)-supported, then $F_1^t(S_{\hat{\Gamma}},d_{\hat{\Gamma}}) \geq F_1^t(S_\Gamma,d_\Gamma)$.

(ii) If $F^t(S_{\hat{\Gamma}},d_{\hat{\Gamma}})$ is unfavourably $(u_1,k \circ u_2)$-supported and k is an increasing function, then $F_1^t(S_{\hat{\Gamma}},d_{\hat{\Gamma}}) \leq F_1^t(S_\Gamma,d_\Gamma)$.

5. RISK SENSITIVITY OF THE BARGAINING SOLUTIONS OF KALAI-ROSENTHAL AND KALAI-SMORODINSKY

The bargaining solution $H\colon BG \to \mathbb{R}^2$, introduced by Kalai and Rosenthal (1978), assigns to $(S,d) \in BG$ the unique element in $[d,i(S)] \cap P(S)$. Here $i(S) := (\max_{s \in S} s_1, \max_{s \in S} s_2)$ is the utopia point of S, and $[d,i(S)]$ is the line segment with end points d and $i(S)$.

The bargaining solution $G\colon BG \to \mathbb{R}^2$, introduced by Kalai and Smorodinsky (1975), assigns to $(S,d) \in BG$ the unique element in $[d,i(S,d)] \cap P(S)$, where $i(S,d)$ is the utopia point of $\{s \in S;\ s \geq d\}$.

In Kihlstrom, Roth and Schmeidler (1980) it was proved that G is risk sensitive on $BSC \cap BSI$. Inspired by this result we show now:

THEOREM 5.1. G and H are risk sensitive on BSR.

PROOF. We only prove that H is risk sensitive on BSR. A similar proof can be given for G. Let $\Gamma = \langle A,\underline{a},u_1,u_2\rangle \in BSR$, $\hat{\Gamma} = \langle A,\underline{a},u_1,k \circ u_2\rangle \in BS$, where k is non-decreasing and concave. We have to show that $H_1(\hat{\Gamma}) \geq H_1(\Gamma)$.
In the following we write $d = (d_1,d_2)$ for d_Γ. Then $d_{\hat{\Gamma}} = (d_1,k(d_2))$. In view of theorem 3.4 we suppose that $d \notin W(S_\Gamma)$. Let $\pi\colon [\alpha,\beta] \to \mathbb{R}$

be the Pareto function of S_Γ and $\hat{\pi}$: $[\hat{\alpha},\beta] \to \mathbb{R}$ the Pareto function of $S_{\hat{\Gamma}}$ $(\hat{\alpha} \geq \alpha)$. For Γ we introduce the function r: $(\max(\alpha,d_1),\beta] \to \mathbb{R}$, defined by

$$r(x) = \pi(x)-(d_2+(x-d_1)(i_1-d_1)^{-1}(i_2-d_2))$$

and for $\hat{\Gamma}$ the corresponding function $\hat{r}$: $(\max(\hat{\alpha},d_1),\beta] \to \mathbb{R}$ with

$$\hat{r}(x) = k(\pi(x))-(k(d_2)+(x-d_1)(i_1-d_1)^{-1}(k(i_2)-k(d_2)).$$

Note that r and $\hat{r}$ are decreasing functions and that the unique points where the function values are zero, are $H_1(\Gamma)$ and $H_1(\hat{\Gamma})$, respectively. Put $x^* = H_1(\Gamma)$. If $x^* \leq \hat{\alpha}$, then $H_1(\hat{\Gamma}) \geq \hat{\alpha} \geq H_1(\Gamma)$. Suppose from now on that $x^* > \hat{\alpha}$. Since $r(x^*) = 0$, we have

$$\pi(x^*) = \alpha i_2+(1-\alpha)d_2 \text{ with } \alpha := (x^*-d_1)(i_1-d_1)^{-1} \in (0,1] \quad (5.1)$$

Now $\hat{r}(x^*) = k(\pi(x^*))-(\alpha k(i_2)+(1-\alpha)k(d_2))$.
From (5.1) and the concavity of k we may conclude that $\hat{r}(x^*) \geq 0$. Since $\hat{r}$ is decreasing, it follows from $\hat{r}(H_1(\hat{\Gamma})) = 0 \leq \hat{r}(x^*)$, that $H_1(\hat{\Gamma}) \geq x^* = H_1(\Gamma)$. □

For risky Pareto outcomes we have the following result. For a proof, see Peters (1981).

THEOREM 5.2. Let $\Gamma = \langle A,\underline{a},u_1,u_2\rangle \in BS$, $\hat{\Gamma} = \langle A,\underline{a},u_1,k \circ u_2\rangle \in BS$ with k non-decreasing and concave. Suppose that $H(S_\Gamma,d_\Gamma)$ is favourably (u_1,u_2)-supported. Then $H_1(S_{\hat{\Gamma}},d_{\hat{\Gamma}}) \geq H_1(S_\Gamma,d_\Gamma)$.

Theorem 4.4(i) and theorem 5.2 are analogous results for different bargaining solutions. A similar result does not hold for the Kalai-Smorodinski solution as the following example shows. The reason is that the two endpoints of $\{x \in P(S);\ x \geq d\}$ are not necessarily utility pairs of riskless outcomes.

EXAMPLE 5.3. Let $\Gamma = \langle A,\underline{a},u_1,u_2\rangle$ where
$A := \mathrm{conv}\{(0,0),(3,-2),(\frac{3}{2},1),(1,\frac{3}{2}),(-1,\frac{5}{2})\}$, $\underline{a} = (0,0)$, $u_1(a_1,a_2) = a_1$, and $u_2(a_1,a_2) = a_2$ for all $(a_1,a_2) \in A$.
Let k and k^* be the increasing concave functions defined by
$k(x) = 2x$ if $x \leq 0$, $k(x) = x$ if $x \geq 0$
$k^*(x) = x$ if $x \leq \frac{3}{2}$ and $k^*(x) = \frac{1}{4}x + \frac{9}{8}$ if $x \geq \frac{3}{2}$.
Let $\hat{\Gamma} = \langle A,\underline{a},u_1,k \circ u_2\rangle$, $\Gamma^* = \langle A,\underline{a},u_1,k^* \circ u_2\rangle$. Then $G_1(\hat{\Gamma}) < G_1(\Gamma)$ and $G_1(\Gamma^*) > G_1(\Gamma)$.

6. SOME CONCLUDING REMARKS

All bargaining solutions we studied until now, proved to be risk sensitive on the class BSR of bargaining situations. The bargaining solutions introduced by Yu (1973) are not risk sensitive on BSR. This can easily be seen when we note that a risk sensitive bargaining solution on BSR is also independent of equivalent utility representations (cf. Roth (1979) p.47) and the Yu-solutions do not have this property. The question arose, whether, conversely, each bargaining solution, which is independent of equivalent utility representations is also risk sensitive on BSR. The answer is no, as the following example shows.

EXAMPLE 6.1. Let $L: BG \to \mathbb{R}^2$ be the bargaining solution with $L(S,d) = F^o(S,d)$, if the Pareto function π_S is a strictly concave function and $L(S,d) = F^1(S,d)$, otherwise. Then L is independent of equivalent utility representations, but not risk sensitive. Take $\Gamma = \langle A,\underline{a},u_1,u_2\rangle$, $\hat{\Gamma} = \langle A,\underline{a},u_1,\sqrt{u_2}\rangle$ with $A = \text{conv}\{(1,0),(0,1),(0,0)\}$, $\underline{a} = (0,0)$, $u_1(x_1,x_2) = x_1$ and $u_2(x_1,x_2) = x_2$ for all $(x_1,x_2) \in A$. Then $L(\Gamma) = (1,0)$, $L(\hat{\Gamma}) = (0,1)$. So $L_1(\hat{\Gamma}) = 0 < 1 = L_1(\Gamma)$.

Note that the bargaining solution L in the above example is not continuous on $\{(S,d) \in BG;\ d \notin W(S)\}$.

Let us conclude this paper with the following conjecture.

CONJECTURE 6.2. Let $\phi: BG \to \mathbb{R}^2$ be a bargaining solution, which is continuous on $\{(S,d) \in BG;\ d \notin W(S)\}$ and which has the independence of equivalent utility representations property. Then the corresponding bargaining solution $\tilde{\phi}: BS \to \mathbb{R}^2$ is risk sensitive on BSR.

REFERENCES

K.J. Arrow, Essays in the Theory of Risk Bearing (Markham, Chicago, 1971).

P.C. Fishburn, Expected utility theories: a review note, in R. Henn and O. Moeschlin, eds., Mathematical Economics and Game Theory (Springer, Berlin, 1977).

J.C. Harsanyi and R. Selten, A Generalized Nash Solution for Two-Person Bargaining Games with Incomplete Information, Management Science 18 (1972) 80-106.

M.J.M. Jansen and S.H. Tijs, Continuity of Bargaining Solutions, Report 8007, Department of Mathematics, Catholic University, Nijmegen, The Netherlands (1980).

E. Kalai, Nonsymmetric Nash Solutions and Replications of 2-Person Bargaining, Intern. J. Game Theory 6 (1977) 129-133.

E. Kalai and R.W. Rosenthal, Arbitration of Two-Party Disputes under Ignorance, Intern. J. Game Theory 7 (1978) 65-72.

E. Kalai and M. Smorodinsky, Other solutions to Nash's bargaining problem, Econometrica 43(1975) 513-518.

R.E. Kihlstrom and L.J. Mirman, Risk Aversion with Many Commodities, J. Econ.Theory 8 (1974) 361-388.

R.E. Kihlstrom, A.E. Roth and D. Schmeidler, Risk Aversion and Solutions to Nash's Bargaining Problem , Working Paper No.37-80, Foerder Institute for Economic Research, Tel-Aviv University, Ramat Aviv, Israel (1980).

J.F. Nash, The Bargaining Problem, Econometrica 18 (1950) 155-162.

H.J.M. Peters, Risk Sensitivity of Bargaining Solutions (in Dutch), Master's Thesis, Department of Mathematics, Catholic University, Nijmegen, The Netherlands (1981).

J.W. Pratt, Risk Aversion in the Small and in the Large, Econometrica 32 (1964) 122-136.

A.E. Roth, Axiomatic Models of Bargaining (Springer, Berlin, 1979).

A.E. Roth and U.G. Rothblum, Risk Aversion and Nash's Solution for Bargaining Games with Risky Outcomes, 1980, to appear in Econometrica.

M.E. Yaari, Some Remarks on Measures of Risk Aversion and on Their Uses, J. Econ. Theory 1 (1969) 315-329.

P.L. Yu, A Class of Solutions for Group Decision Problems, Management Science 19 (1973) 936-946.